ME/BME 646: Mechanics and Control of Human Movement
Winter 2004, Professor Art Kuo
Final Project

Modeling the Leg Motion of a Rower During the Drive and Recovery Phases of the Rowing Stroke

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Abstract

The objective of this project was to study power output during a rowing stroke in order to better understand the effect of timing on boat speed. Rowing coaches place a great deal of emphasis on timing or the synchronization of rowers since it has been observed that a boat with "perfect timing" is fast. But how critical is timing to boat speed and does perfect synchronization really maximize power output and thus boat speed? The goal of this project was to understand how power output was affected by timing however the inability to find the steady-state knee torque required to drive the rowing motion prevented these questions from being addressed. The legs of a rower were modeled as a slider-crank mechanism and the equation of motion governing leg motion during a rowing stroke was derived using Lagrange's equations. MATLAB codes were developed to numerically simulate the equations of motion but all attempts to find the steady-state knee torque curve needed to drive the system failed. Thus the remainder of the report suggests methods for finding the knee torque curve and discusses the type of analysis which should be performed to gain insight into the question of timing and power output.

Introduction

The rowing stroke is a periodic motion comprised of 4 phases: catch, drive, finish, and recovery (Figure 1). Initiation of the drive phase begins at the catch where the blade of the oar is "dropped" into the water and used to "pry" to rowing shell forward. During the drive phase power is generated primarily by the legs and transmitted to the oar via the spine and arms. Thus to examine the energy and power generated during the drive phase equations of motion governing the legs of a rower were derived and simulated numerically. Through simulation of the leg motion one can learn how energy and power vary throughout the rowing stroke. Also the effect of timing on power output can be examined to better understand the relationship between rower synchronization and maximum boat speed.

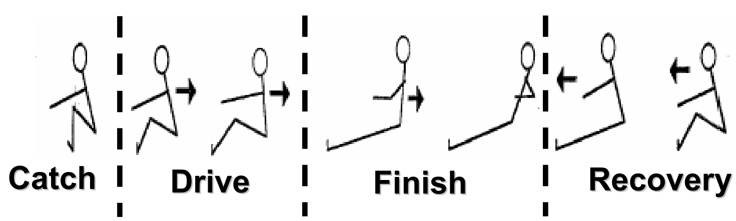
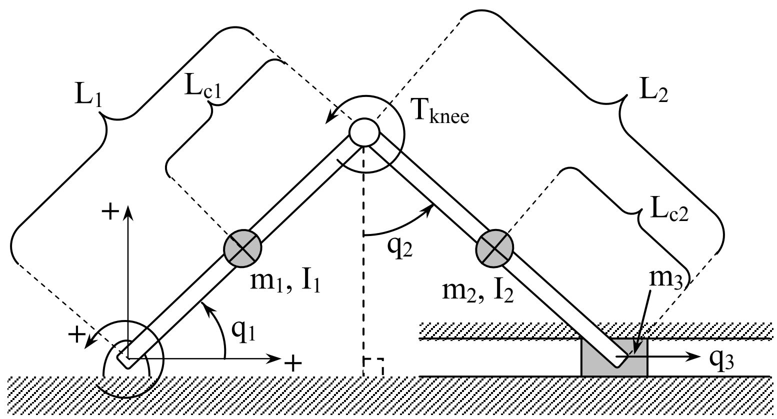


Figure 1: The 4 phases of the rowing stroke.

Model Description

The rower was modeled as a crank-slider mechanism with segments 1 and 2 representing the shank and thigh respectively and the ankle represented by the pin joint. The mass of the upper body (torso, arms, neck, & head) was assumed to be concentrated at the hip and is represented by the slider.



q_1: shank angular displacement
q_2: thigh angular displacement
q_3: upper body translational displacement
m_1: mass of shank
m_2: mass of thigh
m_3: mass of torso, arms, head, & neck
T_{knee}: applied knee torque
L_1: length of shank
L_2: length of thigh
L_{c1}: com. location of shank from proximal end
L_{c2}: com. location of thigh from proximal end
I_1: moment of inertia of shank about com.
I_2: moment of inertia of thigh about com.

Figure 2: 2-Segment Crank-Slider Model of Legs of Rower

Model parameters were assigned numerical values using anthropometric data for a 50% male. The actual values used are presented in Tables II and III in appendix A.

Lagrange's equations were used to derive the equations of motion of the system for the three defined coordinates q_1, q_2, and q_3.

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q'_j}T\right) - \frac{\partial}{\partial q_j}T + \frac{\partial}{\partial q_j}V + \sum_{k=1}^{3}\left(\frac{\partial}{\partial q_j}\phi_k\right)\cdot\lambda_k = Q_j \qquad \text{for } j = 1, 2, 3$$

The kinetic energy of the system, T, is comprised of the translational and rotational kinetic energies of each of the three masses. Gravitational potential energy is the only contribution to the total potential energy of the system, V. Expressions for both T and V are as follows:

$$T = \left(\frac{1}{2}\cdot m_1\cdot v_1^2 + \frac{1}{2}\cdot I_1\cdot \omega_1^2\right) + \left(\frac{1}{2}\cdot m_2\cdot v_2^2 + \frac{1}{2}\cdot I_2\cdot \omega_2^2\right) + \frac{1}{2}\cdot m_3\cdot v_3^2$$

$$V = g\cdot\left(m_1\cdot y_1 + m_2\cdot y_2 + m_3\cdot y_3\right)$$

v_1, v_2, and v_3 denote translational velocities of each of the respective masses and the angular velocities of each segment are given by ω_1 and ω_2 where

$$\omega_1 = q'_1 \quad \text{and} \quad \omega_2 = q'_2$$

The height of each mass relative to ground is given by y_j for j = 1, 2, 3 and is used to compute each masses contribution to the gravitational potential energy of the system.

The slider, m_3, is constrained to move in the x-direction and thus only one of the 3 defined coordinates (q_1, q_2, q_3) is needed to completely define the configuration of the system the crank-slider mechanism is a one degree-of-freedom system. Two constraint equations exist which relate the three coordinates and allow for Lagrange's equations to be reduced to a single ordinary differential equation.

$$\phi_1 = L_1 \cdot \sin(q_1) - L_2 \cdot \cos(q_2) = 0 \qquad \phi_2 = q_3 - L_1 \cdot \cos(q_1) - L_2 \cdot \sin(q_2) = 0$$

First- and second-order derivatives of the constraint equations yield relationships between the velocities and accelerations of the system. These relationships are used in a "weighting scheme" to eliminate the Lagrange multipliers (λ_1, λ_2) and obtain a single equation of motion in terms of the angular acceleration and velocity of q_1 and the angular displacements q_1, and q_2.

$$\begin{pmatrix} q'_2 \\ q'_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cdot q'_1 \qquad \begin{pmatrix} q''_2 \\ q''_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \cdot {q'_1}^2 + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot q''_1$$

Through this process a single differential equation of the following form was obtained:

$$M(q_1, q_2) \cdot q''_1 + B(q_1, q_2) \cdot {q'_1}^2 - F(q_1, q_2) = 0$$

Expressions for $M(q_1,q_2)$, $B(q_1,q_2)$, and $F(q_1,q_2)$ are provided in Appendix B.

Methods

MATLAB files were created to numerically simulate leg motion during a rowing stroke. A list of filenames and a brief description indicating the function of each file is provided in Appendix C along with a copy of the files. A time-varying knee torque was applied to the crank-slider model to drive the system.

<u>Model Input: Knee Torque Curve</u>

The model is driven by an applied knee torque represented by equal and opposite torques acting about the center of mass of the thigh and shank with magnitude ½ T_{knee}.

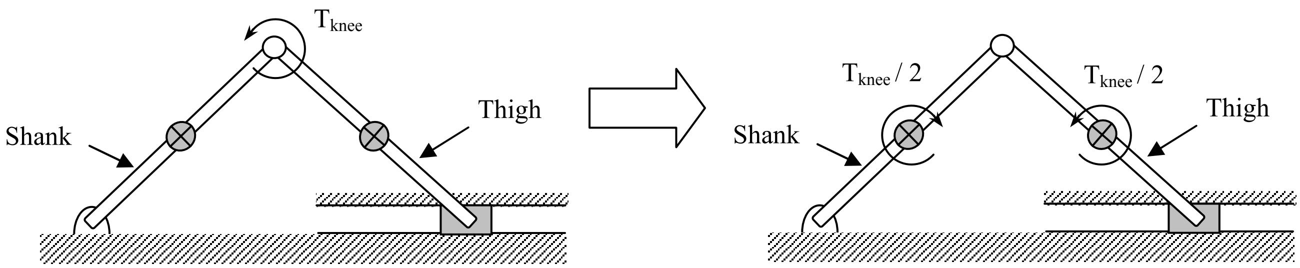


Figure 3: Application of knee torque to slider-crank model

The knee torque is responsible for leg extension during the drive phase and knee flexion during the recovery phase thus the magnitude and direction of the applied knee torque vary throughout the rowing stroke. The difficulty comes in trying to find a knee torque curve which produces leg motion representative of that observed during a rowing stroke. An appropriate knee torque curve should produce the following outcomes:

1) Steady-state behavior of the slider-crank mechanism (i.e. the system returns to its initial state after each stroke)
2) Variation in knee angle throughout the rowing stroke that is consistent with the following published result (where the dotted line is the desired curve)

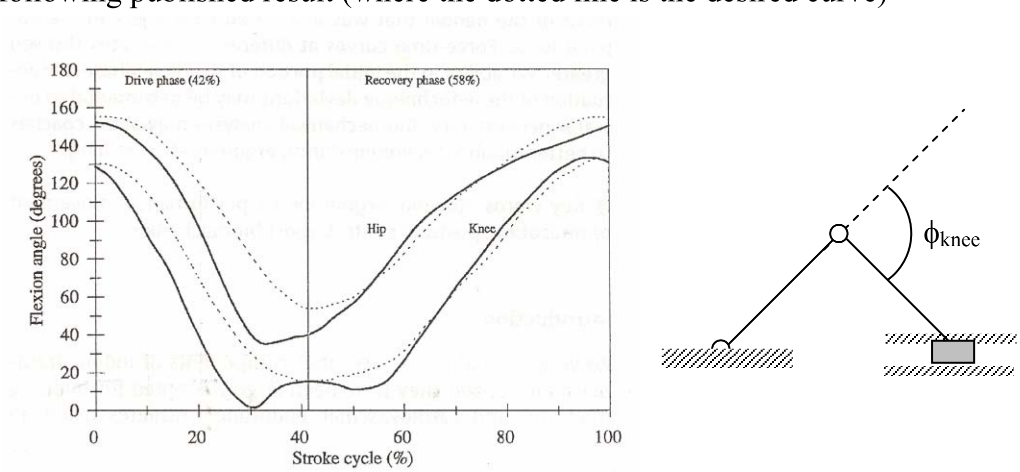


Figure 4: Variation in knee angle, ϕ_{knee}, throughout the rowing stroke

Depending on the type of analysis being performed two different methods were used when trying to find an appropriate knee torque curve.

Transient Analysis: A Single Rowing Stroke

An attempt was made to first analyze the rowing stroke as a single event during which the rower is assumed to start from rest at the catch position. Based on this assumption the shank has zero initial angular velocity and the initial angular displacements are defined by the posture at the catch. An initial shank angular displacement typical of the catch position was obtained from published data. The remaining initial angular displacements were then determined from $q_1(0)$ using the constraint equations.

$q_{1dot}(0)$	0 m/s
$q_1(0)$	66.6 degrees
$q_2(0)$	26.57 degrees
$q_3(0)$	20.65 degrees

Table I: Initial Conditions for Transient Analysis

Exact values of the initial conditions used during this analysis are summarized in Table I.

An applied knee torque acts to extend the legs until the knee reaches near full extension, reverses the direction of motion at the finish by initiating knee flexion, and returns the rower to the initial catch position where the system comes to rest. The difficult part of this analysis method is in finding a knee torque curve which results in this behavior.

Given that rowing is a periodic motion whose period is defined by the stroke rate (i.e. number of strokes / minute) at that an initial knee torque must be applied to start the system an attempt was made to find a torque curve for the transient case using a trial and error approach. Based on the period of the stroke, duration of the drive phase, and nonzero applied knee torque at time t = 0 the knee torque curve was 1st approximated by the linear ramp function illustrated in Figure 5.

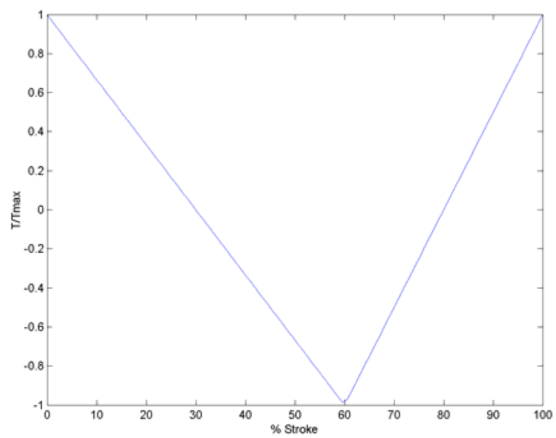


Figure 5: Initial guess for knee torque curve for transient case

This torque curve was modified until the variation in knee angle throughout the rowing stroke was approximately the same as in figure 4.

<u>Steady-State Analysis: Repetitive Rowing Motion</u>

Steady-state analysis refers to the uninterrupted simulation of numerous strokes and requires that both a steady-state knee torque curve and its corresponding fixed-point (i.e. initial conditions which produce steady-state behavior) be identified. An attempt was made to find a steady-state knee torque curve and corresponding fixed-point by using the knee torque curve for the transient case as a starting point. A set of MATLAB codes were created to search for a fixed-point and steady-state knee torque by following the path outlined in figure 6. The general procedure is as follows:

1) Guess values for initial conditions and defining parameters of the knee torque curve.
2) For a given knee torque curve iterate on the initial conditions until a fixed-point is found.
3) Compare the maximum included knee angle occurring during this steady-state simulation with the desired maximum included knee angle of ~164^{o} to determine if the result is physically possible (i.e. knee does not go beyond full extension).

4) If the knee went past full extension then iterate on the parameters of the knee torque curve and repeat the above steps for the newly defined torque curve.
5) Repeat this process until a fixed-point and corresponding knee torque curve are found which produce the desired steady-state behavior.

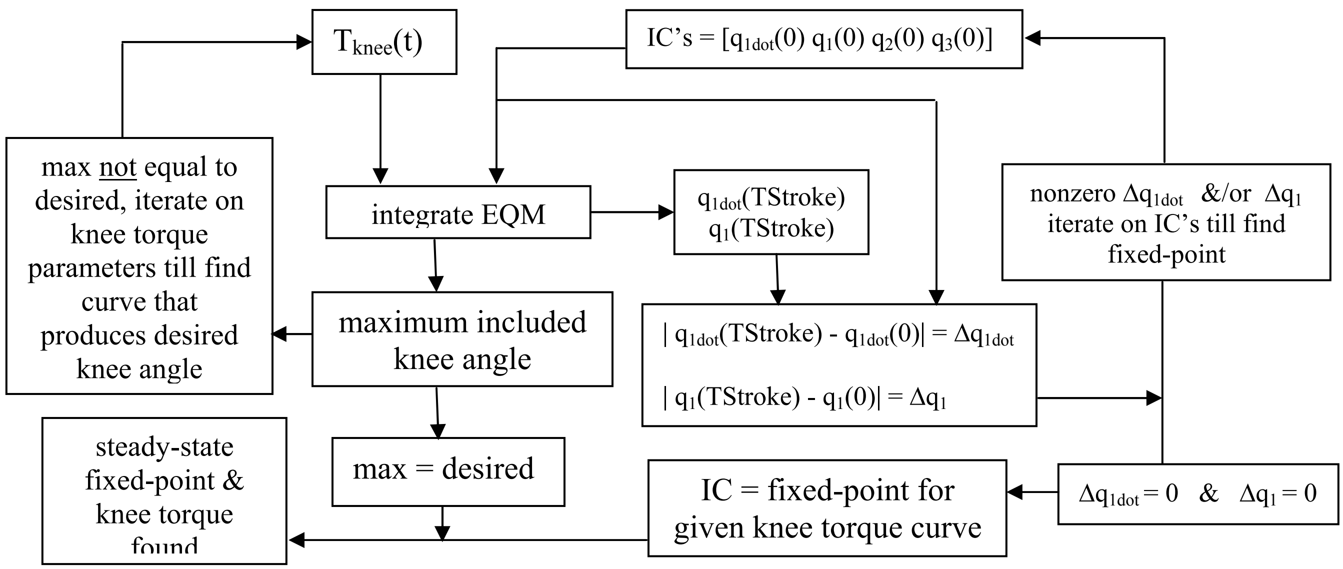


Figure 6: Flow chart of search method used to identify a fixed-point and knee torque curve.

In theory this methodology can be used to obtain a fixed-point and knee torque curve for the steady-state case however the search may fail to converge if the initial conditions and knee torque curve used to initiate the search are not "good" guesses.

Energy and Power Calculations

MATLAB codes **Energy.m** and **StrokePower.m** were written to quantify the variation in energy and power output during the rowing stroke. **Energy.m** computes the kinetic, potential, and total energies associated with each mass as well as energies of the entire system.

Power output at the slider was assumed to be representative of the power provided to the oar and was computed by the function **StrokePower.m**. The velocity of the slider was obtained from the velocity of the shank by using the following relationship given by differentiating the constraint

$$q'_3 = A_2 \cdot q'_1$$

equations. The velocity of the slider was then differentiated numerically to obtain the acceleration and ultimately the force acting at the slider. Power output at the slider was then calculated by multiplying the force times the velocity. The result is the variation in power throughout the duration of a rowing stroke.

Analyzing the Effect of Timing

Given the variation in power over the course of a rowing stroke the effect of timing on power output can easily be determined by simply introducing a phase difference into to the power curve and summing the resulting curves. By varying the phase difference used different timing schemes or degrees of synchronization can be examined to gain insight into the relationship between timing and boat speed.

Results

The following results were generated using a version of the state-derivative function **q.m** which was later found to contain mistakes in the state-space form of the equations of motion. Thus the results presented are not quantitatively correct. However, it took a great deal of time to find the knee torque curve required to produce the single rowing stroke shown in these results and given the time constraints of this project there was not sufficient time to find a knee torque curve for the corrected set of codes. Furthermore, these results, although incorrect, are valuable since they serve as evidence that the problem can be analyzed using the methodology outline above. The difficulty lies with finding an appropriate knee torque curve and, for the case of steady-state, fixed-point which yield the desired behavior.

All of the below results were generated using a stroke rate of 20 strokes per minute which is equivalent to saying that it takes 3 seconds to complete one stroke.

Variation in Model Displacements and Configuration during a Single Rowing Stroke

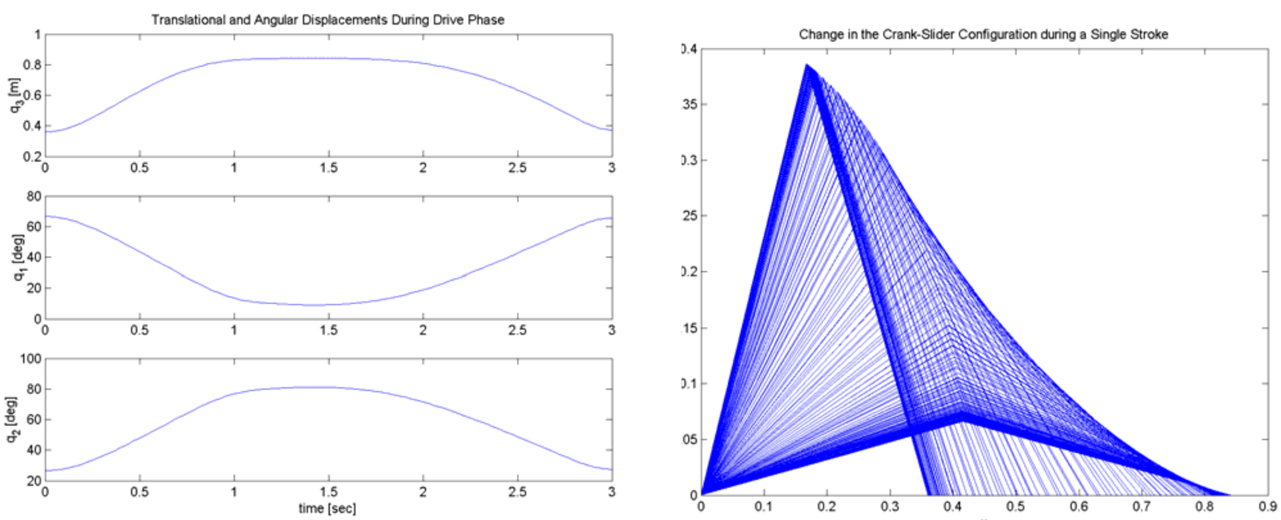


Figure 7: Variation in q1, q2, q3, and the configuration of the model during a stroke.

Knee Torque Curve & Resulting Variation in Knee Angle throughout the Stroke

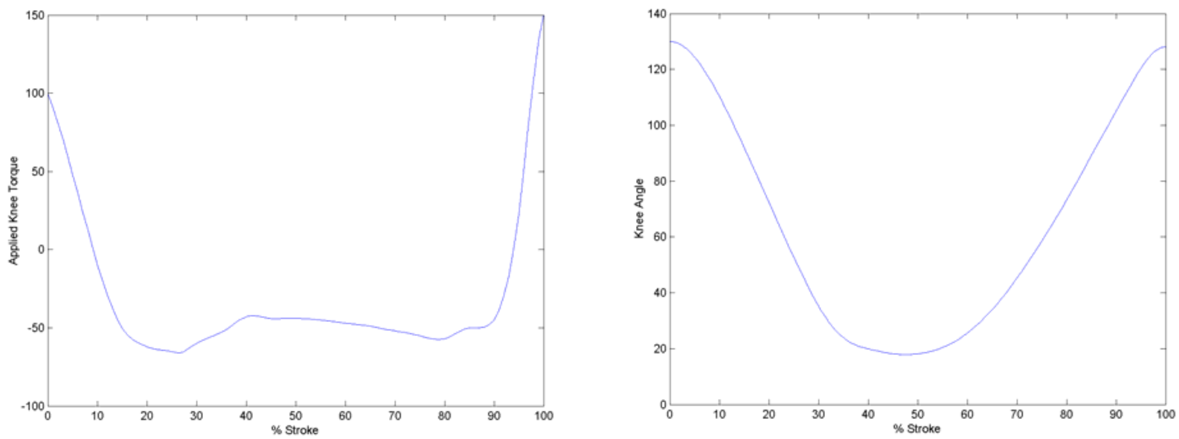


Figure 8: Knee torque curve required to produce a single stroke & the resulting knee angle

Energy and Power Variations during the Rowing Stroke

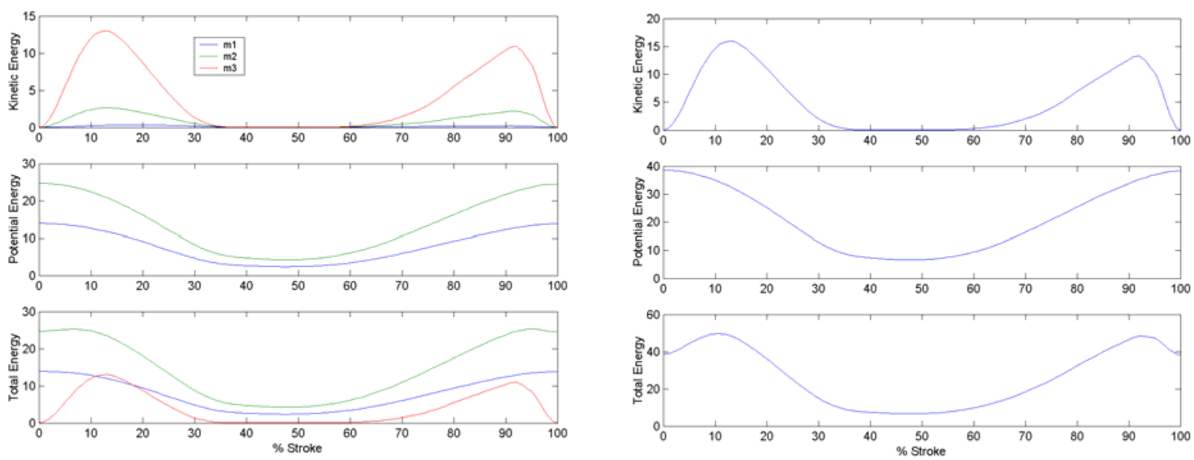


Figure 9: Kinetic, potential, and total energies associated with m_1, m_2, and m_3 and the entire system during a stroke.

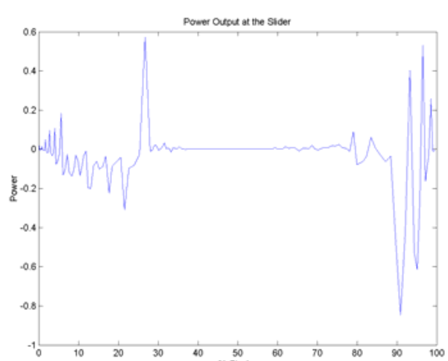


Figure 10: Variation in power output at the slider during a stroke

Discussion

Failure to find a knee torque curve capable of producing either a single stroke or repetitive rowing motion for the corrected set of codes prevented any conclusions from being made regarding the relationship between timing and power output. However, several general observations can be made regarding the proposed methodology based on the results presented above.

Figures 7 and 8 indicate that it is possible to find a knee torque capable of producing leg motion representative of that observed during rowing. Figure 7 shows the leg extending and then flexing such that the knee approaches full extension but does not go beyond the physical limitations of the knee before switching from extension to flexion. Furthermore, figure 8 shows that the applied knee torque can be chosen such that the knee angle is $\sim 130^{o}$ at the initiation *and* completion of the stroke as desired.

In regards to energy, figure 9 indicates that the slider contributes the most to the kinetic energy of the system while the thigh is the largest source of potential energy. It is logical to expect the slider to have the greatest amount of kinetic energy associated with it since the mass of the slider is significantly greater than the mass of the shank or thigh. Figure 9 also indicates that the thigh is the greatest source of energy throughout the stroke which seems reasonable from a rower's perspective since it is well known that the quadriceps are the main driving force in rowing.

Lastly, figure 10 shows how power output at the slider varies throughout the rowing stroke. Power is found to reach its maximum at around 30% of the stroke which corresponds to the end of the drive phase and thus seems reasonable. In general power is found to have the greatest magnitude near the beginning and end of the stroke which is not surprising since this is when a change in the direction of motion is occurring and when inertial effects are significant.

In conclusion, failure to find an appropriate knee torque curve prevented simulation of the rowing motion using the corrected set of MATLAB codes and thus no conclusions were drawn regarding the relationship between timing and power output. However, although finding a knee torque curve which yields the desired behavior is quite difficult, the results obtained earlier, although incorrect, clearly indicate that it is possible. Thus given knowledge of the knee torque curve, either through experiments or some other means, the relationship between timing and power output can be studied using the approach outlined in the methods section.

References

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APPENDIX A: Definition of Model Parameters & Anthropometric Data

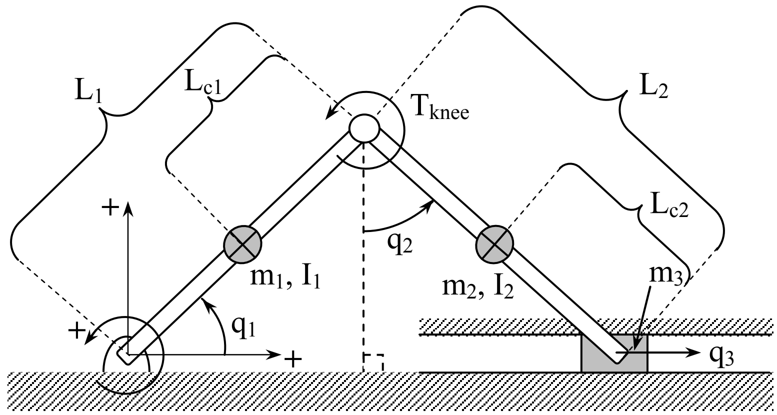

Figure 11: 2-Segment Crank-Slider Model of Legs of Rower

Segment	Segment Wt / Total Body Wt	COM / Segment Length	Radius of Gyration / Segment Length
		Proximal	*Center of Gravity*
Shank	0.043	0.433	0.302
Thigh	0.100	0.433	0.323
Head, neck, trunk & arms	0.686	N/A	N/A

Table II: Anthropometric Data (Chaffin, et al., *Occupational Biomechanics, 3rd edition*)

Segment	Mass, m_i [kg]	Length, L_i [m]	COM, Lc_i [m]	Moment of Inertia (COM), I_i [kg-m2]
2 x Shank (i = 1)	6.47	0.421	0.182	0.1045
2 x Thigh (i = 2)	15.0	0.432	0.187	0.2928
Torso (i = 3)	51.6	N/A	N/A	N/A

Table III: Anthropometric Measures of a 50% Male (Body Mass = 75.2 kg)

APPENDIX B: Equations of Motion for Crank-Slider Model

Kinematic Relationships

(x, y) coordinates defining the position of m_1, m_2, and m_3

$x_1 = \alpha \cdot \cos(q_1)$ $\qquad$ $y_1 = \alpha \cdot \sin(q_1)$ $\qquad$ where, $(L_1 - L_{c1}) = \alpha$

$x_2 = L_1 \cdot \cos(q_1) + \beta \cdot \sin(q_2)$ $\qquad$ $y_2 = L_1 \cdot \sin(q_1) - \beta \cdot \cos(q_2)$ $\qquad$ $(L_2 - L_{c2}) = \beta$

$x_3 = q_3$ $\qquad$ $y_3 = 0$

x & y velocity components of m_1, m_2, and m_3

$x'_1 = -\alpha \cdot \sin(q_1) \cdot q'_1$ $\qquad$ $y'_1 = \alpha \cdot \cos(q_1) \cdot q'_1$

$x'_2 = -L_1 \cdot \sin(q_1) \cdot q'_1 + \beta \cdot \cos(q_2) \cdot q'_2$ $\qquad$ $y'_2 = L_1 \cdot \cos(q_1) \cdot q'_1 + \beta \cdot \sin(q_2) \cdot q'_2$

$x'_3 = q'_3$ $\qquad$ $y'_3 = 0$

square of resultant velocity for m_1, m_2, and m_3

$$v_1^2 = {x'_1}^2 + {y'_1}^2 = \alpha^2 \cdot {q'_1}^2$$

$$v_2^2 = {x'_2}^2 + {y'_2}^2 = L_1^2 \cdot {q'_1}^2 + \beta^2 \cdot {q'_2}^2 + 2 \cdot L_1 \cdot \beta \cdot q'_1 \cdot q'_2 \cdot \sin(q_2 - q_1)$$

$$v_3^2 = {q'_3}^2$$

Lagrange's Equations

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q'_1} T\right) - \frac{\partial}{\partial q_1} T + \frac{\partial}{\partial q_1} V + \left(\frac{\partial}{\partial q_1} \phi_1\right) \cdot \lambda_1 + \left(\frac{\partial}{\partial q_1} \phi_2\right) \cdot \lambda_2 = Q_1$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q'_2} T\right) - \frac{\partial}{\partial q_2} T + \frac{\partial}{\partial q_2} V + \left(\frac{\partial}{\partial q_2} \phi_1\right) \cdot \lambda_1 + \left(\frac{\partial}{\partial q_2} \phi_2\right) \cdot \lambda_2 = Q_2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial q'_3} T\right) - \frac{\partial}{\partial q_3} T + \frac{\partial}{\partial q_3} V + \left(\frac{\partial}{\partial q_3} \phi_1\right) \cdot \lambda_1 + \left(\frac{\partial}{\partial q_3} \phi_2\right) \cdot \lambda_2 = Q_3$$

Kinetic Energy of the System

$$T = \left(\frac{1}{2}\cdot m_1\cdot v_1^2 + \frac{1}{2}\cdot I_1\cdot \omega_1^2\right) + \left(\frac{1}{2}\cdot m_2\cdot v_2^2 + \frac{1}{2}\cdot I_2\cdot \omega_2^2\right) + \frac{1}{2}\cdot m_3\cdot v_3^2$$

Potential Energy of the System

$V = g\cdot(m_1\cdot y_1 + m_2\cdot y_2 + m_3\cdot y_3)$ where, $y_1 = \alpha\cdot\sin(q_1)$ $y_2 = L_1\cdot\sin(q_1) - \beta\cdot\cos(q_2)$ $y_3 = 0$

Constraint Equations

$\phi_1 = L_1\cdot\sin(q_1) - L_2\cdot\cos(q_2) = 0$ $\phi_2 = q_3 - L_1\cdot\cos(q_1) - L_2\cdot\sin(q_2) = 0$

Lagrange's Equations in Matrix Form

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\cdot\begin{pmatrix} q''_1 \\ q''_2 \\ q''_3 \end{pmatrix} + \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \\ G_{31} & G_{32} \end{pmatrix}\cdot\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$

where,

$M_{11} = m_1\cdot\alpha^2 + I_1 + m_2\cdot L_1^2$

$M_{12} = M_{21} = m_2\cdot L_1\cdot\beta\cdot\sin(q_2 - q_1)$

$M_{13} = M_{31} = M_{23} = M_{32} = 0$

$M_{22} = m_2\cdot\beta^2 + I_2$

$M_{33} = m_3$

$G_{11} = L_1\cdot\cos(q_1)$

$G_{12} = L_1\cdot\sin(q_1)$

$G_{21} = L_2\cdot\sin(q_2)$

$G_{22} = -L_2\cdot\cos(q_2)$

$G_{31} = 0$

$G_{32} = 1$

$F_1 = Q_1 - m_2\cdot L_1\cdot\beta\cdot q'^2_2\cdot\cos(q_1 - q_2) - g\cdot(m_1\cdot\alpha + m_2\cdot L_1)\cdot\cos(q_1)$

$F_2 = Q_2 + m_2\cdot L_1\cdot\beta\cdot q'^2_1\cdot\cos(q_1 - q_2) - g\cdot m_2\cdot\beta\cdot\sin(q_2)$

$F_3 = Q_3$

Velocity & Acceleration Relationships from Differentiating Constraint Equations

$\begin{pmatrix} q'_2 \\ q'_3 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\cdot q'_1$ where, $A_1 = \dfrac{-L_1\cdot\cos(q_1)}{L_2\cdot\sin(q_2)}$ $A_2 = \dfrac{-L_1\cdot\cos(q_2)\cdot\cos(q_1)}{\sin(q_2)} - L_1\cdot\sin(q_1)$

and

$$\begin{pmatrix} q''_2 \\ q''_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \cdot q'_1{}^2 + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \cdot q''_1 \quad \text{where,} \quad B_1 = \frac{L_1 \cdot \sin(q_1)}{L_2 \cdot \sin(q_2)} - \frac{\cos(q_2)}{\sin(q_2)} \cdot A_1{}^2 \qquad C_1 = A_1$$

$$B_2 = -L_1 \cdot \cos(q_1) - L_2 \cdot \sin(q_2) \cdot A_1{}^2 + L_2 \cdot \cos(q_2) \cdot B_1 \qquad C_2 = -L_1 \cdot \sin(q_1) + L_2 \cdot \cos(q_2) \cdot C_1$$

Eliminate Lagrange Multipliers and Reduce System of Equations to Single ODE

$$1 \cdot (M_{11} \cdot q''_1 + M_{12} \cdot q''_2 + G_{11} \cdot \lambda_1 + G_{12} \cdot \lambda_2 = F_1)$$

$$A_1 \cdot (M_{21} \cdot q''_1 + M_{22} \cdot q''_2 + G_{21} \cdot \lambda_1 + G_{22} \cdot \lambda_2 = F_2)$$

$$+ \quad A_2 \cdot (M_{33} \cdot q''_3 + G_{32} \cdot \lambda_2 = F_3)$$

$$(M_{11} + A_1 \cdot M_{21}) \cdot q''_1 + (M_{12} + A_1 \cdot M_{22}) \cdot q''_2 + A_2 \cdot M_{33} \cdot q''_3 + (G_{11} + A_1 \cdot G_{21}) \cdot \lambda_1 + (G_{12} + A_1 \cdot G_{22} + A_2 \cdot G_{32}) \cdot \lambda_2 = F_1 + A_1 \cdot F_2 + A_2 \cdot F_3$$

where,

$$G_{11} + A_1 \cdot G_{21} = L_1 \cdot \cos(q_1) + \left(\frac{-L_1 \cdot \cos(q_1)}{L_2 \cdot \sin(q_2)} \right) \cdot L_2 \cdot \sin(q_2) = 0$$

$$G_{12} + A_1 \cdot G_{22} + A_2 \cdot G_{32} = L_1 \cdot \sin(q_1) + \left(\frac{-L_1 \cdot \cos(q_1)}{L_2 \cdot \sin(q_2)} \right) \cdot (-L_2 \cdot \cos(q_2)) + \left[-L_1 \cdot \cos(q_2) \cdot \left(\frac{\cos(q_1)}{\sin(q_2)} \right) - L_1 \cdot \sin(q_1) \right] = 0$$

$$q''_2 = B_1 \cdot q'_1{}^2 + C_1 \cdot q''_1$$

$$q''_3 = B_2 \cdot q'_1{}^2 + C_2 \cdot q''_1$$

thus, after substituting, the above equation of motion reduces to the following form

$$M(q_1, q_2) \cdot q''_1 + B(q_1, q_2) \cdot q'_1{}^2 - F(q_1, q_2) = 0$$

where,

$$M(q_1, q_2) = M_{11} + A_1 \cdot M_{21} + C_1 \cdot (M_{12} + A_1 \cdot M_{22}) + C_2 \cdot A_2 \cdot M_{33}$$

$$B(q_1, q_2) = (M_{12} + A_1 \cdot M_{22}) \cdot B_1 + A_2 \cdot M_{33} \cdot B_2$$

$$F(q_1, q_2) = F_1 + A_1 \cdot F_2 + A_2 \cdot F_3$$

State-Derivative for Simulation of Governing Equation of Motion

$$y \equiv \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} q'_1 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \qquad y' \equiv \begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \\ y'_4 \end{pmatrix} = \begin{pmatrix} q''_1 \\ q'_1 \\ q'_2 \\ q'_3 \end{pmatrix} = \begin{pmatrix} F(y_2, y_3) - B(y_2, y_3) \cdot y_1^{\,2} \\ y_1 \\ A_1 \cdot y_1 \\ A_2 \cdot y_1 \end{pmatrix}$$

APPENDIX C: MATLAB Codes

CrankSlider.m: defines system parameters, runs simulation, calls Energy.m and StrokePower.m: to compute energies and power output throughout stroke

q.m: state-derivative function

KneeTorque.m: defines knee torque curve

Energy.m: computes energies of m1, m2, m3, and the entire system as a whole

StrokePower.m: computes power output at the slider

Rower.m: runs search methodology outlined in figure 6 to determine fixed-point and knee torque curve for steady-state case

SteadyStateIC.m: iterates on IC until steady-state is achieved for a given torque curve

Knee.m: iterates on knee torque curve parameters until the maximum included knee angle occurring during the stroke is ~164^{o}