HW #4 - SOLUTIONS

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Problem 3.5

Given:

$$w = 1200 \text{ lbs}, \quad E = 30 \times 10^6 \text{ psi},$$

$$I = 10 \text{ in.}^4, \quad L = 8 \text{ ft}; \quad \zeta = 1\%$$

$$p_o = 60 \text{ lbs}; \quad \omega = \left(\frac{300}{60}\right) 2\pi = 10\pi \text{rads/sec}$$

Stiffness of two beams:

$$k = 2\left(\frac{48EI}{L^3}\right) = 32{,}552 \text{ lbs/in.}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{w/g}} = \sqrt{\frac{32{,}552}{1200/386}} = 102.3 \text{ rads/sec}$$

Steady state response:

$$R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

where $\omega/\omega_n = 10\pi/102.3 = 0.3071$. Therefore,

$$R_d = \frac{1}{\sqrt{[1 - 0.0943]^2 + [2 \times 0.01 \times 0.3071]^2}} = 1.104$$

Displacement:

$$u_o = (u_{st})_o R_d = \frac{p_o}{k} R_d$$

$$= \frac{60}{32{,}552} \times 1.104 = 2.035 \times 10^{-3} \text{ in.}$$

Acceleration amplitude:

$$\ddot{u}_o = \omega^2 u_o = (10\pi)^2 \, 2.035 \times 10^{-3}$$

$$= 2.009 \text{ in./sec}^2 = 0.0052\text{g}$$

Problem 3.6

In Eq. (3.2.1) replacing the applied force by $p_o \cos\omega t$ and dividing by m we get

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = \frac{p_o}{m}\cos\omega t \qquad \text{(a)}$$

(a) The particular solution is of the form:

$$u_p(t) = C\sin\omega t + D\cos\omega t \qquad \text{(b)}$$

Differentiating once and then twice gives

$$\dot{u}_p(t) = C\omega\cos\omega t - D\omega\sin\omega t \qquad \text{(c)}$$

$$\ddot{u}_p(t) = -C\omega^2\sin\omega t - D\omega^2\cos\omega t \qquad \text{(d)}$$

Substituting Eqs. (b)-(d) in Eq. (a) and collecting terms:

$$\left[(\omega_n^2 - \omega^2)C - 2\zeta\omega_n\omega D\right]\sin\omega t$$

$$+ \left[2\zeta\omega_n\omega C + (\omega_n^2 - \omega^2)D\right]\cos\omega t = \frac{p_o}{m}\cos\omega t$$

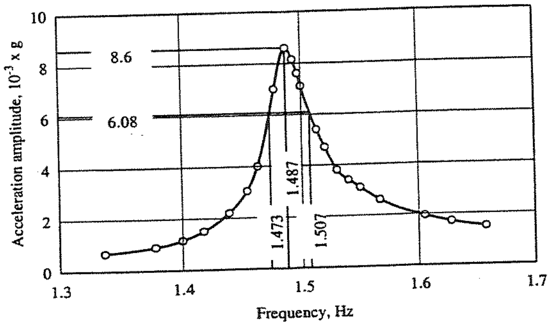
Equating coefficients of $\sin\omega t$ and of $\cos\omega t$ on the two sides of the equation:

$$(\omega_n^2 - \omega^2)C - (2\zeta\omega_n\omega)D = 0 \qquad \text{(e)}$$

$$(2\zeta\omega_n\omega)C + (\omega_n^2 - \omega^2)D = \frac{p_o}{m} \qquad \text{(f)}$$

Problem 3.11

The given data is plotted in the form of the frequency response curve shown in the accompanying figure:



(a) Natural frequency

The frequency response curve peaks at

$$f_n = 1.487 \text{ Hz}$$

Assuming small damping, this value is the natural frequency of the system.

(b) Damping ratio

The acceleration at the peak is $r_{peak} = 8.6 \times 10^{-3}$ g. Draw a horizontal line at $r_{peak} \div \sqrt{2} = 6.08 \times 10^{-3}$ g to obtain f_a and f_b in Hz:

$$f_a = 1.473 \text{ Hz} \qquad f_b = 1.507 \text{ Hz}$$

Then,

$$\zeta = \frac{f_b - f_a}{2 f_n} = \frac{1.507 - 1.473}{2\,(1.487)} = 0.0114$$

$$= 1.14\%$$

Problem 3.18

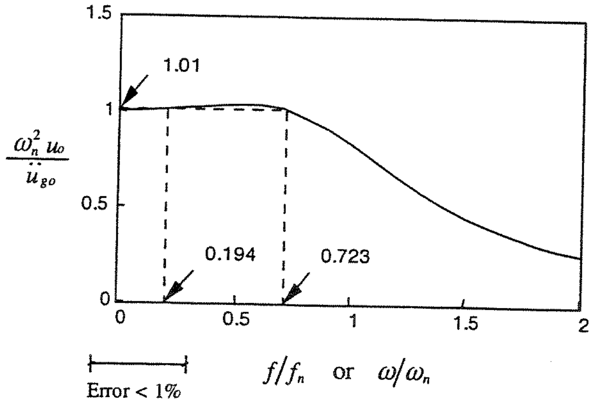
From Eq. (3.7.1),

$$\omega_n^2\, u(t) = -R_d\, \ddot{u}_g(t - \phi/\omega) = -R_d\, \ddot{u}_{go} \sin(2\pi f t - \phi)$$

and therefore,

$$\frac{\omega_n^2 u_o}{\ddot{u}_{go}} = R_d = \frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} \qquad \text{(a)}$$

Eq. (a) is plotted for $\zeta = 0.6$, the accelerometer damping ratio:



We want to bound R_d as follows:

$$0.99 \le R_d \le 1.01 \qquad \text{(b)}$$

The relevant condition is $R_d \le 1.01$ because we are interested in a continuous range of frequencies over which the error is less than 1%. Therefore, impose

$$\frac{1}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}} = 1.01 \qquad \text{(c)}$$

Defining $\beta \equiv \omega/\omega_n$, Eq. (c) can be rewritten as

$$(1 - \beta^2)^2 + (2\zeta\beta)^2 = \left(\frac{1}{1.01}\right)^2 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 1 = 0.9803 \Rightarrow$$

$$\beta^4 - 0.56\beta^2 + 0.0197 = 0 \Rightarrow$$

$$\beta^2 = 0.0377,\ \ 0.5223 \Rightarrow \beta = 0.194,\ \ 0.723$$

Choose $\beta = 0.194$ (see figure) which gives the desired frequency range:

$$f \le 0.194\, f_n = 0.194\,(25) \Rightarrow$$

$$f \le 4.86 \text{ Hz}$$

Problem 3.23

From Eq. (3.8.1),

$$E_D = 2\pi\zeta(\omega/\omega_n)\, k u_o^2 \qquad \text{(a)}$$

where

$$u_o = \frac{p_o}{k} R_d \qquad \text{(b)}$$

In Eq. (a) substituting Eq. (b) and Eq. (3.2.11) for R_d gives

$$E_D = \frac{\pi p_o^2}{k} \frac{2\zeta\omega/\omega_n}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}$$

Problem 3.26

(a) $p(t)$ is an even function:

$$p(t) = p_o\left(1 - \frac{2}{T_0}t\right) \qquad 0 \le t \le T_0/2 \quad \text{(a)}$$

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p(t)dt = \frac{2}{T_0}\int_0^{T_0/2} p_o\left(1 - \frac{2}{T_0}t\right)dt$$

$$\begin{aligned}
a_j &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} p(t)\cos(j\omega_0 t)dt \\
&= \frac{4p_0}{T_0}\int_0^{T_0/2}\left(1-\frac{2}{T_0}t\right)\cos(j\omega_0 t)dt \\
&= \frac{4p_0}{T_0}\left\{\frac{1}{j\omega_0}\left[\sin(j\omega_0 t)\right]_0^{T_0/2} - \frac{2}{T_0}\int_0^{T_0/2} t\cos(j\omega_0 t)dt\right\} \\
&= -\frac{8p_0}{T_0^2}\int_0^{T_0/2} t\cos\left(\frac{2\pi jt}{T_0}\right)dt \\
&= -\frac{4p_0}{\pi jT_0}\left\{\left[t\sin\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2} + \frac{T_0}{2\pi j}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2}\right\} \\
&= -\frac{2p_0}{\pi^2 j^2}\left[\cos\left(\frac{2\pi j}{T_0}t\right)\right]_0^{T_0/2} \\
&= -\frac{2p_0}{\pi^2 j^2}\left[\cos(\pi j) - 1\right]
\end{aligned}$$

$$\therefore a_j = \begin{cases} \dfrac{4p_o}{\pi^2 j^2} & j = 1,3,5,\cdots \\ 0 & j = 2,4,6,\cdots \end{cases} \qquad \text{(c)}$$

$b_n = 0$ because $p(t)$ is an even function (d)

Thus the Fourier series representation of $p(t)$ is

$$p(t) = \frac{p_o}{2} + \frac{4p_o}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2}\cos(j\omega_0 t) \qquad \text{(e)}$$

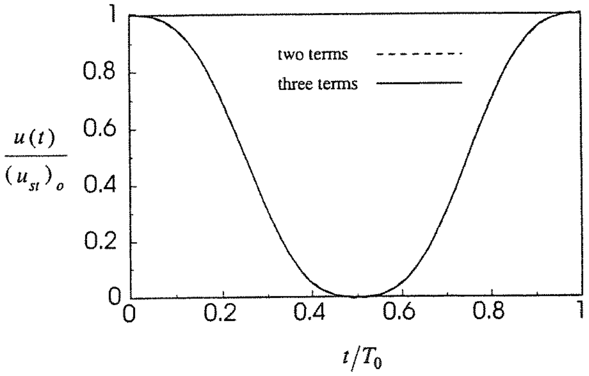
(b) The steady-state response of an undamped system is obtained by substituting Eqs. (b), (c) and (d) in Eq. (3.13.6) to obtain

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{4}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(1-\beta_j^2)}\cos(j\omega_0 t) \qquad \text{(f)}$$

where $(u_{st})_o = p_o/k$ and $\beta_j = j\omega_0/\omega_n$. Equation (f) is indeterminate when $\beta_j = 1$; corresponding values of T_0 are T_n, $3T_n$, $5T_n$, etc.

(c) For $T_0/T_n = 2$, $\beta_j = j\omega_0/\omega_n = jT_n/T_0 = j/2$ and Eq. (f) becomes

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{2} + \frac{16}{\pi^2}\sum_{j=1,3,5,\cdots}^{\infty}\frac{1}{j^2(4-j^2)}\cos\left(\frac{2\pi jt}{T_0}\right)$$



Because of the j^4 in the denominator of the series, two terms are enough to obtain reasonable convergence of the series solution.

Problem 4.5

(a) The equation of motion is

$$m\ddot{u} + ku = p_o e^{-at} \tag{a}$$

Using Duhamel's integral the solution is

$$u(t) = \frac{p_o}{m\omega_n}\int_0^t e^{-a\tau}\sin\left[\omega_n(t-\tau)\right]d\tau \tag{b}$$

Integrate by parts letting $v = \sin\left[\omega_n(t-\tau)\right]$ and $dy = e^{-a\tau}d\tau$:

$$u(t) = vy - \int y dv$$

or

$$u(t) = \frac{p_o}{m\omega_n a}\left\{\left[-e^{-a\tau}\sin\left[\omega_n(t-\tau)\right]\right]_0^t \right.$$
$$\left. - \int_0^t \omega_n e^{-a\tau}\cos\left[\omega_n(t-\tau)\right]d\tau\right\}$$

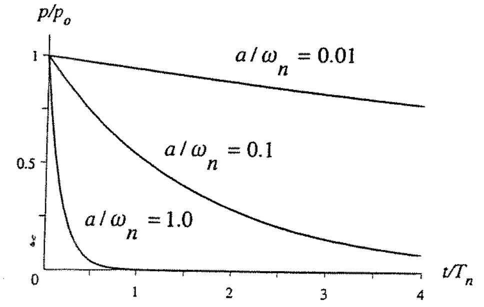
Integrating again by parts, this time with $v = \cos\left[\omega_n(t-\tau)\right]$ and $dy = e^{-a\tau}d\tau$ gives

$$u(t) = \frac{p_o}{m\omega_n(a^2+\omega_n^2)}\left[a\sin\omega_n t - \omega_n\cos\omega_n t + \omega_n e^{-at}\right]$$

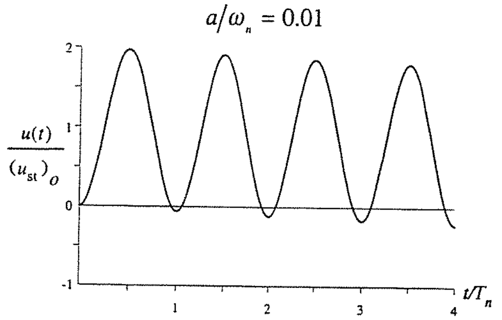
Written in terms of $(u_{st})_o$, the displacement response is

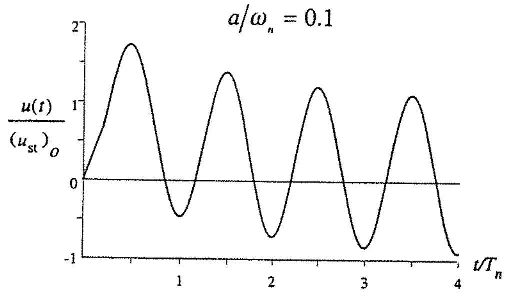
$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1+\frac{a^2}{\omega_n^2}}\left[\frac{a}{\omega_n}\sin\omega_n t - \cos\omega_n t + e^{-at}\right] \tag{c}$$

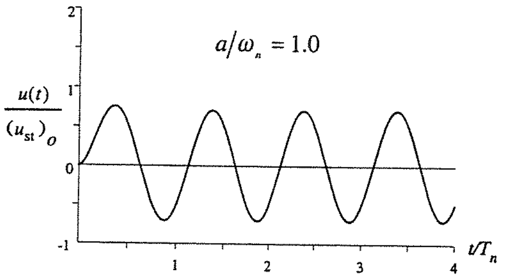
(b) The force $p(t) = p_o e^{-at}$ is plotted for three values of a/ω_n:



The motion given by Eq. (c) is plotted next.







As t increases, term e^{-at} becomes very small and the system attains a steady state harmonic motion whose amplitude is given by

$$\frac{u_{\text{steady}}}{(u_{st})_o} = \frac{1}{\sqrt{1+a^2/\omega_n^2}}$$

Problem 4.13

(a) Response results.

We have from Eq. (4.3.2)

$$\left.\begin{aligned} u(t) &= (u_{\text{st}})_o(1-\cos\omega_n t) \\ \dot u(t) &= (u_{\text{st}})_o\,\omega_n \sin\omega_n t \end{aligned}\right\} \quad 0\le t\le T_n/2 \qquad \text{(a)}$$

At $t=T_n/2$, $u=2(u_{\text{st}})_o$ and $\dot u=0$.

For $T_n/2\le t\le T_n$,

$$u(\bar t) = A_1\cos\omega_n\bar t + A_2\sin\omega_n\bar t - \frac{p_o}{k}$$

where $\bar t = t - T_n/2$. Substituting $u(0)=2(u_{\text{st}})_o$ and $\dot u(0)=0$ gives $A_1=3(u_{\text{st}})_o$ and $A_2=0$. Hence

$$u(t)=(u_{\text{st}})_o[-1+3\cos\omega_n(t-T_n/2)],\ T_n/2\le t\le T_n \quad \text{(b)}$$

At $t=T_n$, i.e. at $\bar t=T_n/2$, $u=-4(u_{\text{st}})_o$ and $\dot u=0$.

For $T_n\le t\le 3T_n/2$

$$u(\bar t) = A_1\cos\omega_n\bar t + A_2\sin\omega_n\bar t + \frac{p_o}{k}$$

where $\bar t = t - T_n$. Substituting $u(0)=-4(u_{\text{st}})_o$ and $\dot u(0)=0$ gives $A_1=5(u_{\text{st}})_o$ and $A_2=0$. Hence

$$u(t)=(u_{\text{st}})_o[1-5\cos\omega_n(t-T_n)]\ ,\ T_n\le t\le 3T_n/2 \quad \text{(c)}$$

At $t=3T_n/2$, i.e. at $\bar t=T_n/2$, $u=6(u_{\text{st}})_o$ and $\dot u=0$.

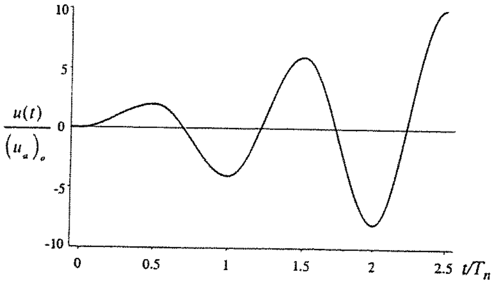
In a similar manner the following results can be obtained:

$$u(t)=(u_{\text{st}})_o[-1+7\cos\omega_n(t-3T_n/2)]\ ,\ 3T_n/2\le t\le 2T_n \quad \text{(d)}$$

$$u(t)=(u_{\text{st}})_o[1-9\cos\omega_n(t-2T_n)]\ ,\ 2T_n< t< 5T_n/2 \quad \text{(e)}$$

(b) Response plot.

From Eqs. (a), (b), (c), (d), and (e), $u(t)/(u_{\text{st}})_o$ is plotted against t/T_n. Note that $\omega_n(t-T_n/2)=2\pi(t/T_n-1/2)$, $\omega_n(t-T_n)=2\pi(t/T_n-1)$, etc.



(c) Peak values.

The displacement peaks u_n at the end of n half cycles of applied force are

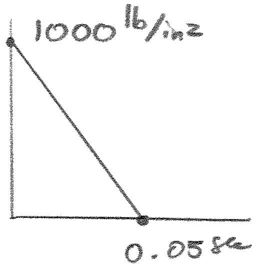
n	1	2	3	4	5
$u_n/(u_{\text{st}})_o$	2	-4	6	-8	10

In general,

$$\frac{u_n}{(u_{\text{st}})_o} = (-1)^{n-1}2n$$

PROBLEM # 6

LOOKS LIKE TRIANGULAR LOAD:



$t_d = 0.05$ sec

$T_n = 0.25$ sec

$\therefore \frac{t_d}{T_n} = 0.2$

USE BIGG'S CHART TO FIND EQUIVALENT STATIC LOAD (DLF = Dynamic Load Factor) × 1000 lb

DLF = 0.6

$\therefore$ Design for $p = 600$ lb/in²